Modelled approximations to the ideal filter with application to time series of Gross Domestic Product

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1 Introduction

Cyclical patterns of expansion-contraction occur in a broad range of time series

Time-varying intensity and with changing duration around some central period.

Example: cycles in economic data, typically recur with periodicity ranging from around 2 to 10 years

Filters: 1) Hodrick-Prescott (1997) filter as low-pass filter, HP trend substracted from series, detrending filter, but this leaves noise in estimated "cycle"

2) Solution: Band-pass filter, remove both low and high frequencies, smoother estimates of the cycle, Baxter and King (1999) and Harvey and Trimbur (2003).

Time series models for cyclical behavior

Dynamics as decomposition into unobserved components

2 Modelling stochastic cycles

Andrews (1994) - looked at forecast performance of STSMs for economic data

Mendelssohn (2005) - models for El Nino related phenomena

Fadiga and Wong (2009), Clark and Coggin (2009) - regional housing prices

Mills (2012) - EMU countries overall output gap

Labys et al (2000) - cycles in commodity prices.

Tawadros (2008) - cyclial dynamics of oil demand

De Bonis and Silvestrini (2014) - financial cycle in Italy.

Successful studies with **first order cycles**, but these....

1. Leave noise in the cycle, worse for turning point analysis
2. Often do worse than higher orders in fitting the data
3. Do not give link with band-pass filter methods

<u>Higher order cycles</u> - solves the three points, less well studied

3 Estimating cycles/filtering

Basic goal of filtering: extract the cyclical component in series that has other components

$$y_t = \mu_t + \psi_t + \varepsilon_t,$$

Implicit or explicit, Either decomposition is spelled out directly or it is implied by filtering

Band pass filtering is to remove μ_t and ε_t

μ_t - Trend: nonstationarity, long-run, low frequencies

ψ_t - Cycle: stochastic, periodic, mid-range frequencies

ε_t - Noise, higher frequencies

4 "Ideal" Filter

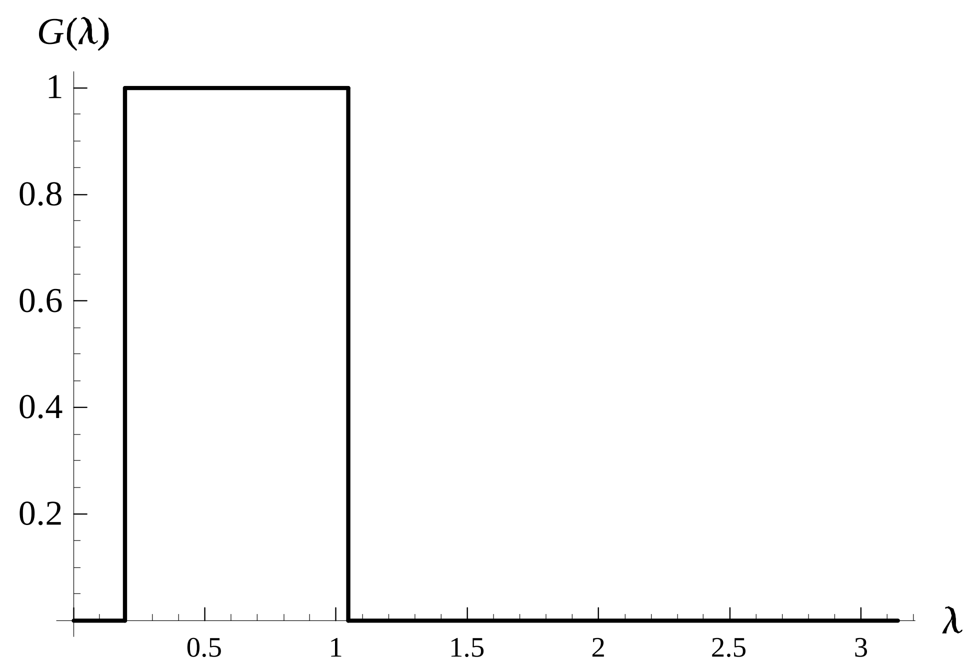
Emulate a certain gain function for the filter, sharp or block-like shape, pass through frequencies within a band without altering amplitudes, completely annihilate frequencies outside the band

Simple concept

1. Does not directly use any information about the series' process, makes very strong assumptions implicitly about process

2. One size fits all

This can create critical problems



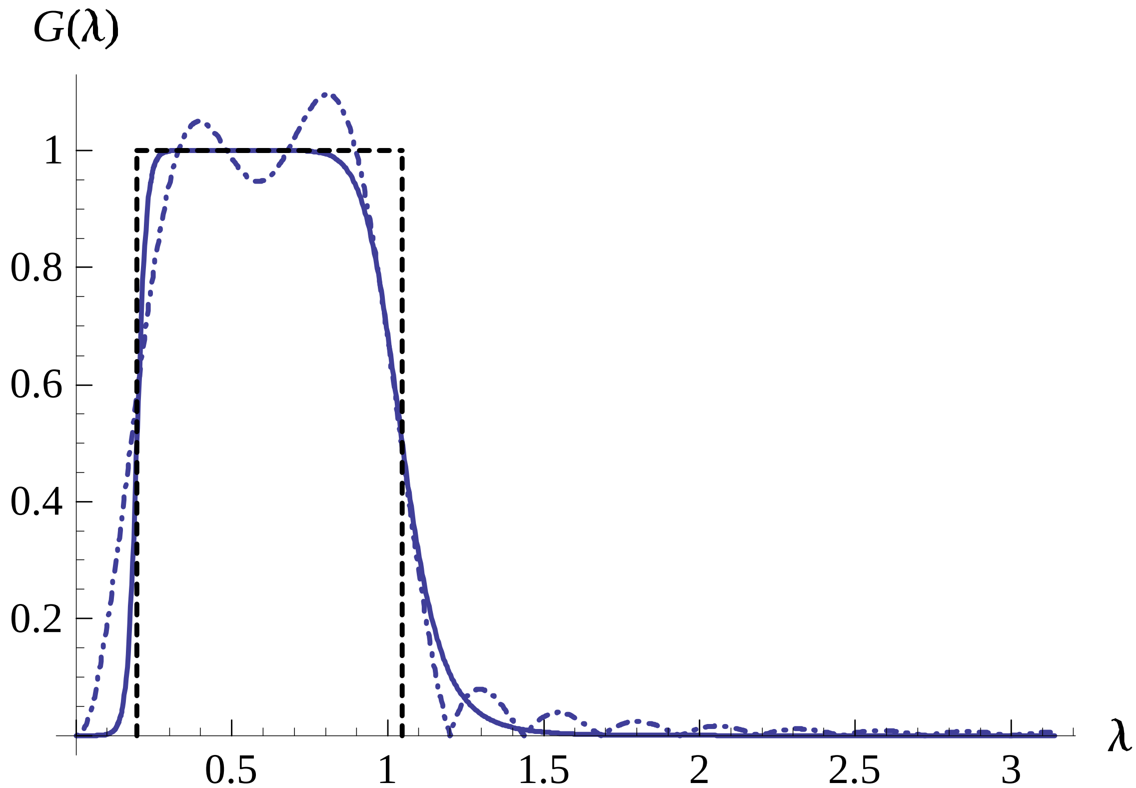
5 Baxter and King (1999): BK Filter

Time invariant, fixed filter, fixed set of weights, automatic computation

Immediate Disadvantages (as approximation to ideal filter): 1. Truncation of estimates at end of sample - the most important ones for policy and current analysis

2. Unstable gain - large ripple over pass band and more oscillations at higher frequencies

BK gain shown in figure below as dotted-dashed line, starting from origin, gain moves to 1.05 then to 0.95, then finally to 1.1 before declining toward zero and showing more ripples



Gain function for BK filter (truncating twelve observations) and modelled representation of ideal filter.

6 Alternative To BK filter is modelled filter

Start with parametric class of gains, generalize the Butterworth Band Pass filter

$$GB^{bp}_{m,n}(\lambda; \lambda_c, \rho, q_\zeta, q_\kappa)$$

$$= \frac{q_\kappa [c(\lambda)]^n}{q_\zeta / [(1+\phi^2 - 2\phi \cos\lambda)(2 - 2\cos\lambda)^{m-1}] + q_\kappa [c(\lambda)]^n + 1},$$

$$c(\lambda) = \frac{1+\rho^2 \cos^2\lambda_c - 2\rho \cos\lambda_c \cos\lambda}{1+\rho^4 + 4\rho^2 \cos^2\lambda_c - 4(\rho+\rho^3)\cos\lambda_c \cos\lambda + 2\rho^2 \cos 2\lambda}$$

positive integers m and n are indices of filter

$q_\zeta > 0$ and $q_\kappa > 0$ are referred to as "signal-noise" ratios

ϕ and ρ mainly determine the filters' width and curvature

λ_c is a major frequency that determines the location of band-pass filter

Offer a lot of flexibility in gain shape

To get modelled "Ideal" Filter - Set parameters to generate sharp band-pass

7 Set of Modelled Representations of Ideal Filter

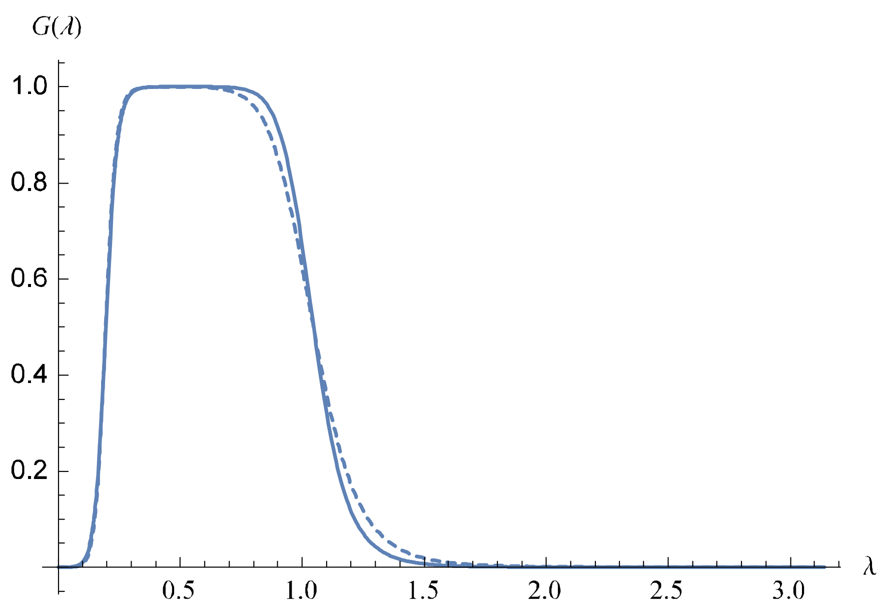
Need moderate to high band pass index, so use $n = 4, 6,$ 8.

Need high damping coefficient and highly persistent trend slope, so use $\rho = 0.8, \phi = 0.97$

For each n, use different parameter settings, while constraining gain to equal 1/2 at edges

Figure below shows gain functions for different approximations with $n = 6$

Solid curve $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{$0.0124, 0.0322, 0.4910$\}$ Dotted curve $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{$2.524, 0.279, 0.398$\}$.

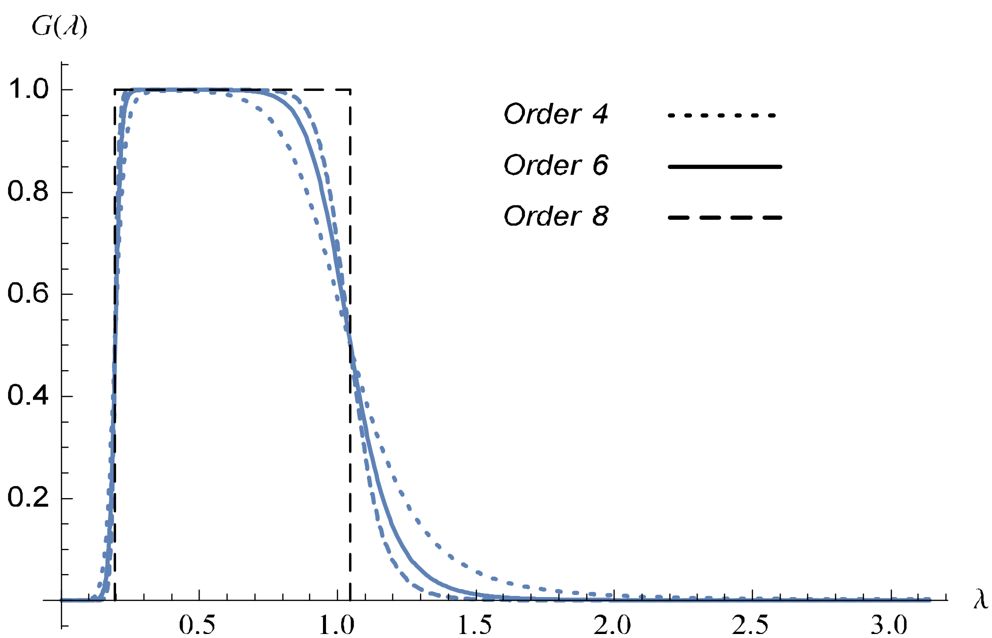


Major finding: Sharpness depends mostly on index n, get tight range (non-trivial but modest) by making large changes to other parameters

8 Set of Modelled Representations of Ideal Filter

$n = 4, 6$, and 8

For each n, pick one of 12 candidate approximations on basis of how well underlying model/parameter values fit across range of time series of economic activity.



Three modelled representations of the ideal filter for various orders.

Main finding: Flexibility in right side of band pass – in noise elimination greater than flexibility in left side – trend or low frequency removal

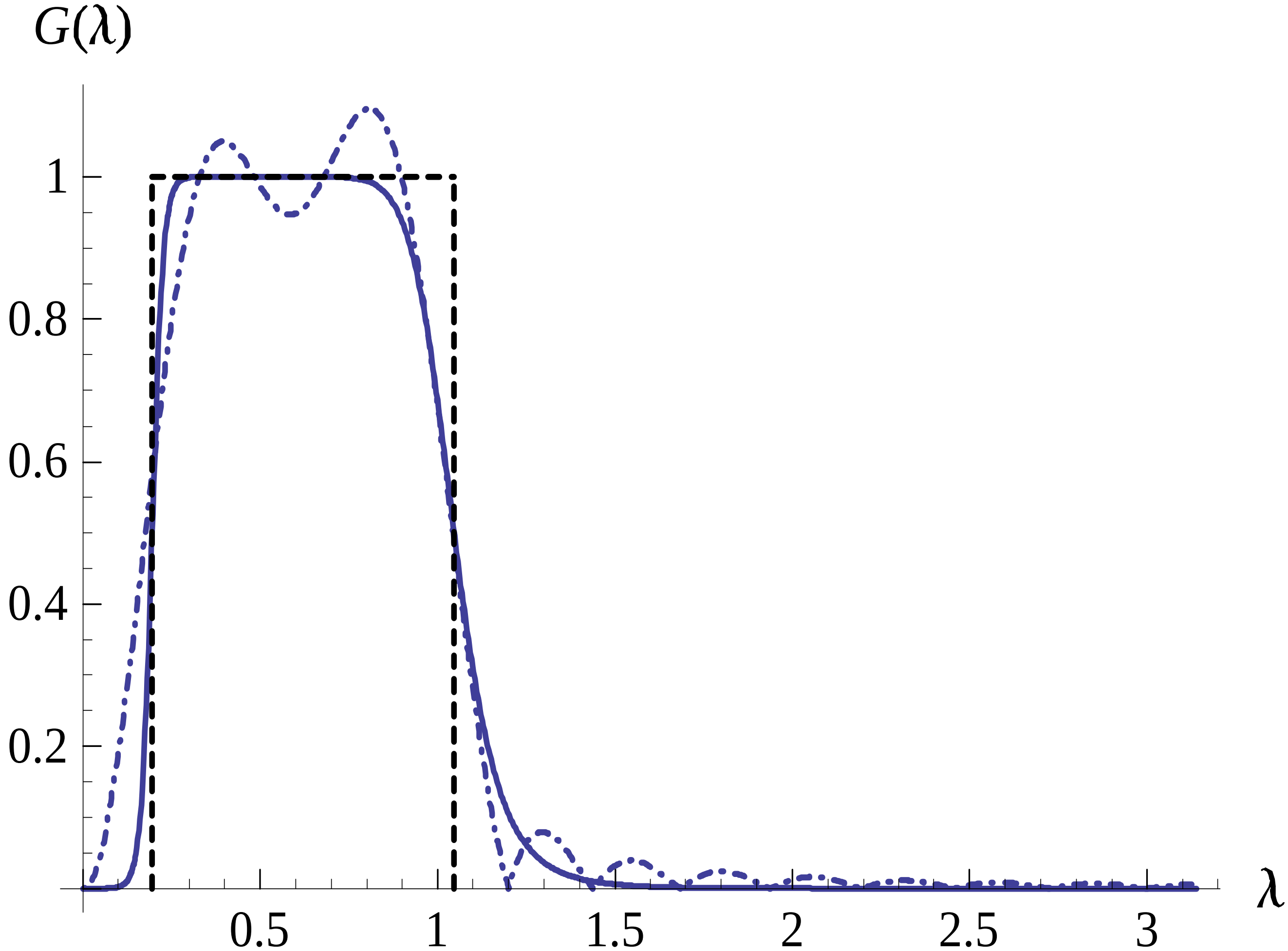
9 Baxter and King (1999): BK Filter

Figure below shows $n = 6$ modelled ideal filter as solid line

Modelled version of ideal filter fixes the two immediate problems with BK

1. Near end point estimates given by finite sample treatment of underlying model

2. Gain increases smoothly and remains very close to one over pass band with very little variability, then decreases monotonically toward zero at higher frequencies



Gain function for BK filter (truncating twelve observations) and modelled representation of ideal filter.

10 Ideal Filter Usage

Ideal Filter Usage:

Offers automatic application

Seems "scientific" but actually involves strong assumptions

What are the crucial shortfalls of the BK filter (or other ideal filter emulator) ?

Artificial cycles can be created from pure noise or trend

More generally, cycles can be inflated or dampened by the filtering procedure – problematic for policy decisions

Basic problem is that gain emulation makes no use of information about process

11 BK filter creates "cycle" from noise

Figure below shows simulated series of constant level plus white noise

Adaptive filter (from Trend-Cycle-Irregular model) vs. BK filter

Estimated trend shown in blue - estimated as approximately constant

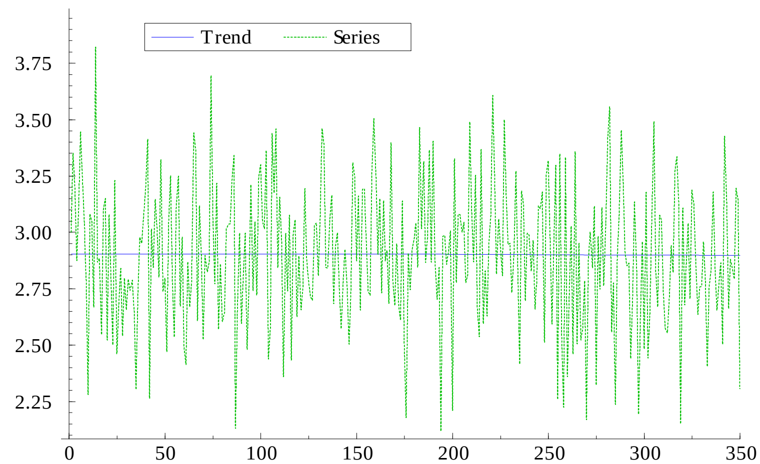
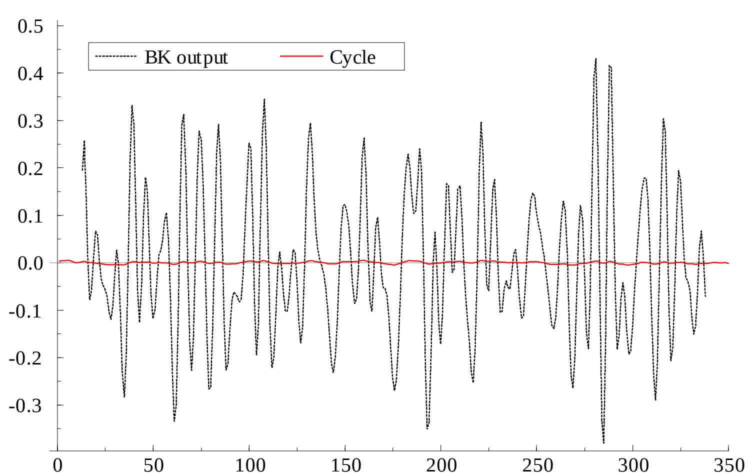


Figure below shows "cycle" from BK output and estimated cycle

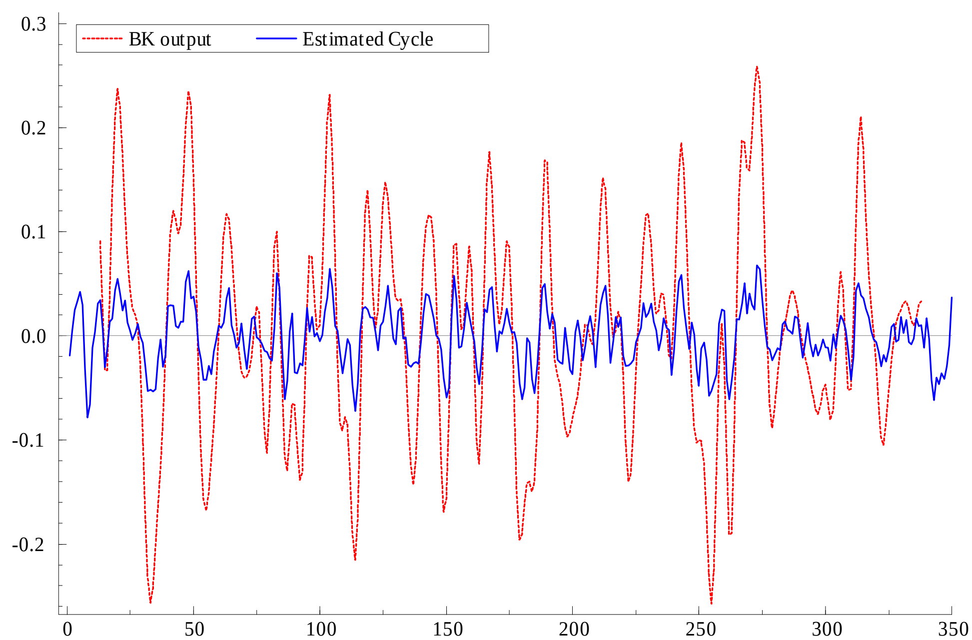
BK output has clear serial correlation – looks like a stochastic cycle with peaks and troughs and rapid oscillations

Estimated cycle is close to zero and correctly so



12 BK filter creates cycle from RW trend

Simulated random walk and estimated trend using model-based approach.



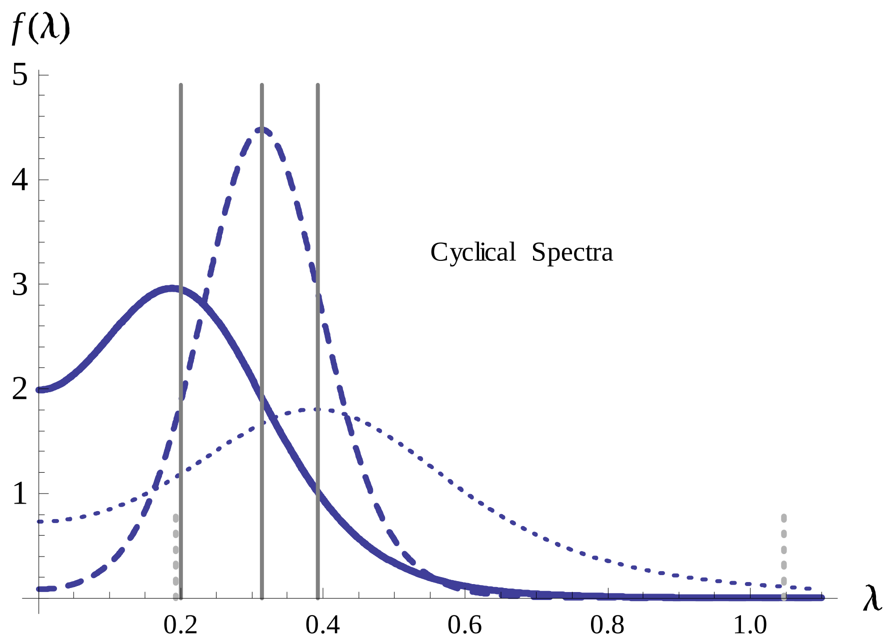
Estimated cycle using model-based approach vs. output from the BK filter, for a simulated random walk.

13 Information about periods applies directly to the cyclical component

Periodicity does NOT apply directly to the filter used to extract it

Baxter and King (1999) explicitly state that Burns and Mitchell (1946) specified business cycles as cyclical components of no less than six quarters in duration and that they typically last fewer than eight years

Also, note the use of "typically last", suggesting possibility of cyclical episodes longer than 8 years, as happened in post-WWII era —> this argues against a sudden and sharp cutoff for periods above 8 years



We use a flexible concept of cycle that has peaked spectrum – a lot of diversity permitted

14 Stochastic Cycle Model

The model is constructed by starting with a simple fixed cycle, writing as difference equation, adding damping, including stochastic shocks, then adding resonance

Start with fixed cycle:

$$\psi_t = A \cos(\lambda_c t + \omega)$$

with period $2\pi/\lambda_c$, amplitude A, and phase ω being fixed parameters

Next, re-express the cycle

$$\psi_t = (\psi_0 \cos \lambda_c t + \psi_0^* \sin \lambda_c t)$$

which is equivalent with $A = \sqrt{(\psi_0^2 + \psi_0^{*2})}$ and $\omega = \tan^{-1}(\psi_0^*/\psi_0)$.

15 Stochastic Cycle Model

Now, convert to linear difference form by augmenting the system with a companion process ψ_t^* that also evolves over time, and then writing a recursion:

$$\begin{bmatrix} \psi_t \\ \psi_t^* \end{bmatrix} = \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{bmatrix} \psi_{t-1} \\ \psi_{t-1}^* \end{bmatrix}$$

where the constants ψ_0 and ψ_0^* become the starting values

Extend with ρ, autoregressive factor where $0 < \rho \leq 1$:

$$\begin{bmatrix} \psi_t \\ \psi_t^* \end{bmatrix} = \rho \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{bmatrix} \psi_{t-1} \\ \psi_{t-1}^* \end{bmatrix}$$

16 Stochastic Cycle Model

Add shocks:

$$\begin{bmatrix} \psi_t \\ \psi_t^* \end{bmatrix} = \rho \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{bmatrix} \psi_{t-1} \\ \psi_{t-1}^* \end{bmatrix} + \begin{bmatrix} \kappa_t \\ \kappa_t^* \end{bmatrix}$$

where $[\kappa_t, \kappa_t^*] \sim WN(0, \sigma_\kappa^2)$ and κ_s is uncorrelated with κ_t^* for all s, t

Then add resonance and define a second order cycle $\psi_{2.t}$ by

$$\begin{bmatrix} \psi_{2,t} \\ \psi_{2,t}^* \end{bmatrix} = \rho \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{bmatrix} \psi_{2,t-1} \\ \psi_{2,t-1}^* \end{bmatrix} + \begin{bmatrix} \psi_{1,t} \\ \psi_{1,t}^* \end{bmatrix}$$

$$\begin{bmatrix} \psi_{1,t} \\ \psi_{1,t}^* \end{bmatrix} = \rho \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{bmatrix} \psi_{1,t-1} \\ \psi_{1,t-1}^* \end{bmatrix} + \begin{bmatrix} \kappa_t \\ \kappa_t^* \end{bmatrix}$$

17 Stochastic Cycle Model

Now add higher orders of resonance and define a general class.

For $n > 0$, an nth order stochastic cycle $\psi_{n,t}$ is given by the following:

$$\begin{bmatrix} \psi_{i,t} \\ \psi^*_{i,t} \end{bmatrix} = \rho \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{bmatrix} \psi_{i,t-1} \\ \psi^*_{i,t-1} \end{bmatrix} + \begin{bmatrix} \psi_{i-1,t-1} \\ \psi^*_{i-1,t-1} \end{bmatrix}$$

for $i = 2, ..., n$

$$\begin{bmatrix} \psi_{1,t} \\ \psi^*_{1,t} \end{bmatrix} = \rho \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{bmatrix} \psi_{1,t-1} \\ \psi^*_{1,t-1} \end{bmatrix} + \begin{bmatrix} \kappa_t \\ \kappa^*_t \end{bmatrix}$$

Higher n gives sharper spectrum

18 Stocastic Trend, Signal Extraction

Consider a damped smooth trend:

$$\mu_t = \mu_{t-1} + \beta_{t-1}, \tag{1}$$
$$\beta_t = (1-\phi)\overline{\beta} + \phi\beta_{t-1} + \zeta_t, \qquad \zeta_t \sim WN(0, \sigma_\zeta^2)$$

where β_t is the slope, coefficient ϕ satisfies $0 < \phi \leq 1$ and gives trend damping.

band pass: best (MMSE) estimator of cyclical component

$$\psi_t = F^{bp}(L; M, \theta) y_t$$

Frequency domain: $F^{bp}_{m,n}(\lambda) = \frac{g_{\psi_n}(\lambda)}{g_{\mu_m}(\lambda) + g_{\psi_n}(\lambda) + g_\varepsilon(\lambda)}$

When you separate cycle from trend and noise, it is intuitive that best filter for the task depends on how cycle relates to trend and noise

19 Signal Extraction - adaptive band-pass

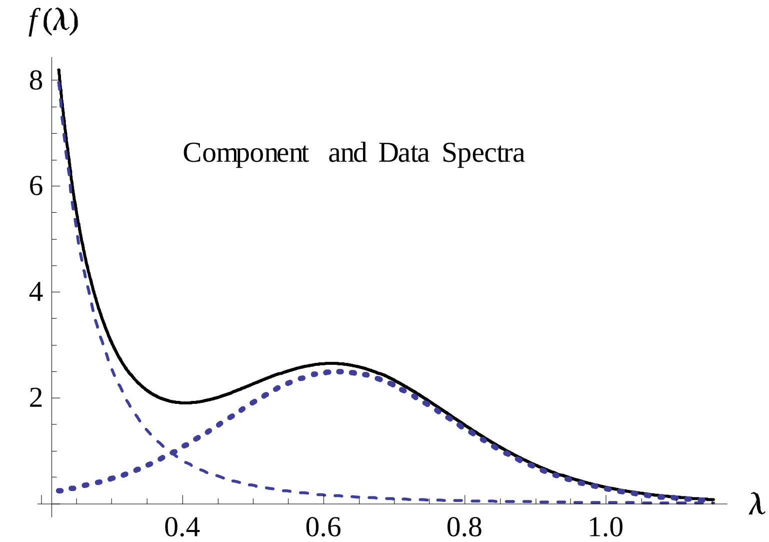


Illustration of trend pseudospectrum and stochastic cycle spectrum.

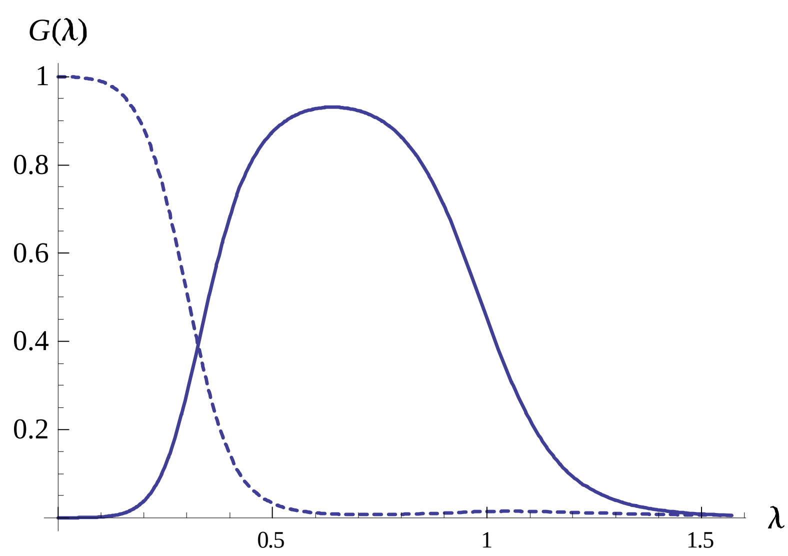


Illustration of trend (low-pass) and cycle (band-pass) filters formed from model.

Curved band-pass, from overlapping frequency mix of components

Shape varies across series (different trend-cycle relationships)

20 Application

Data: Quarterly real GDP and GDP components

12 series in total on major sectors of economic activity

{ "Gross Domestic Product", "Investment", "Residential Investment", "Non-Residential Investment", "Inventory Change", "Consumption", "Consumption of Durables" "Consumption of Non-Durables", "Consumption of Services", "Government Expenditures", "Exports", "Imports"}

Source: Published by Bureau of Economic Analysis in 2018Q1

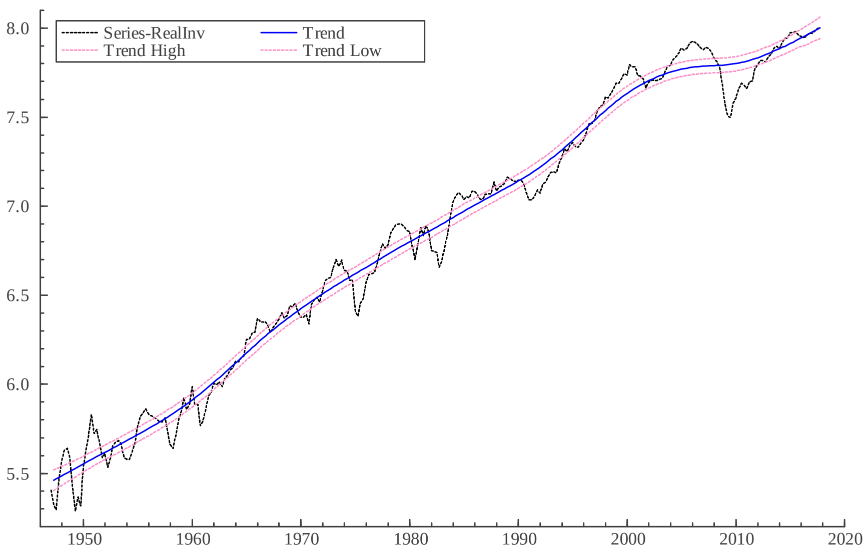
Sample: 1947:1 to 2017:4

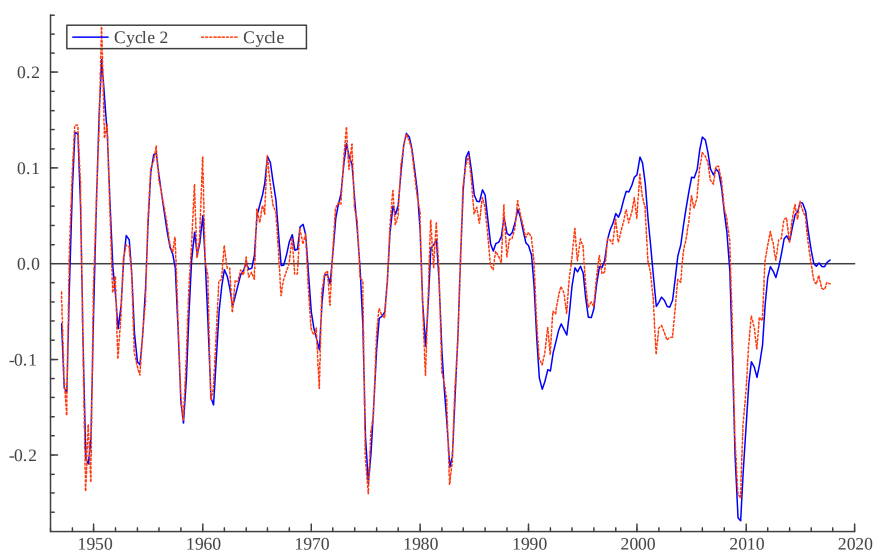
Range of properties in data, various trend-cycle characteristics

Such a dataset has advantage of being able to try method on different trend-cycle scenarios

Example: Investment, trend and confidence bands shown in figure below

slight variation in growth rate up to GR, then trend flattens out during GR, then growth rate recovers some





Estimated cycle in Investment for n = 1 and 2 with Balanced form.

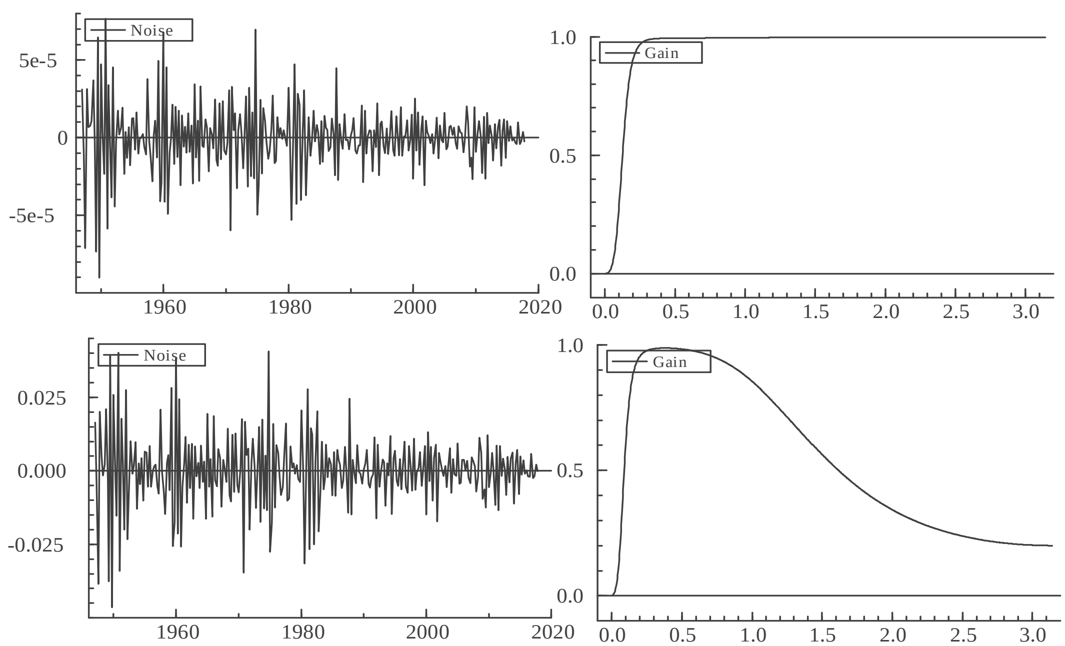
Large difference in smoothness between n = 1 and 2, second order better for turning point analysis and for study of major cyclical movements. $n = 2$ case also fits much better.

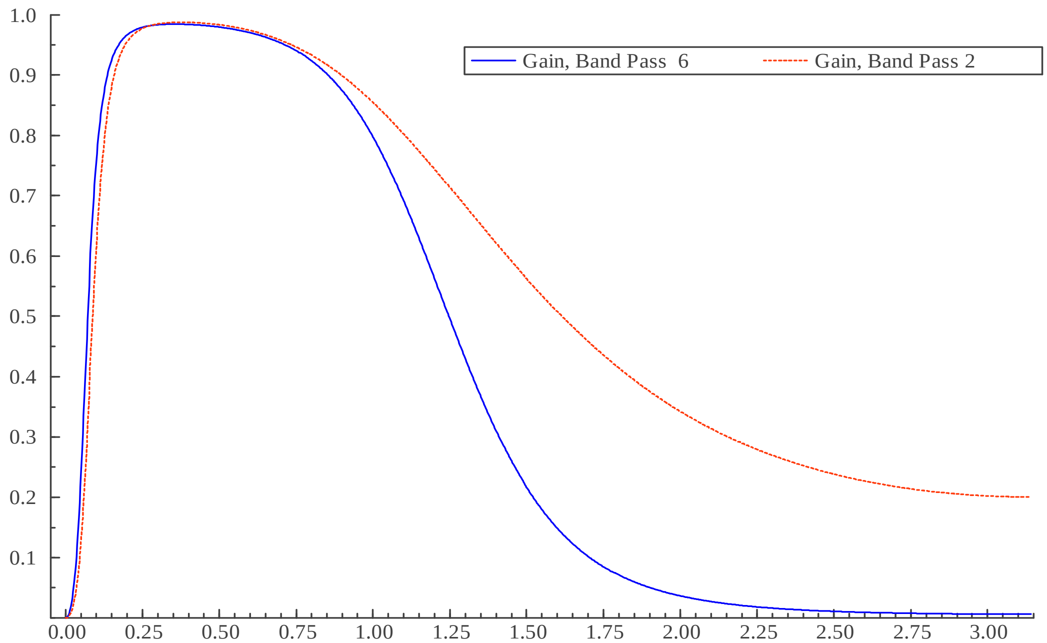
Second order case also gives link with band pass, whereas n = 1 case does not.

Figure below shows noise removed on left and gain for extracting cycle on right. First row gives n = 1, and second row gives n = 2 results.

n = 1 gives high pass filter – no removal of high frequencies

n = 2 shows the right tail characteristic of band pass, removal of high frequencies





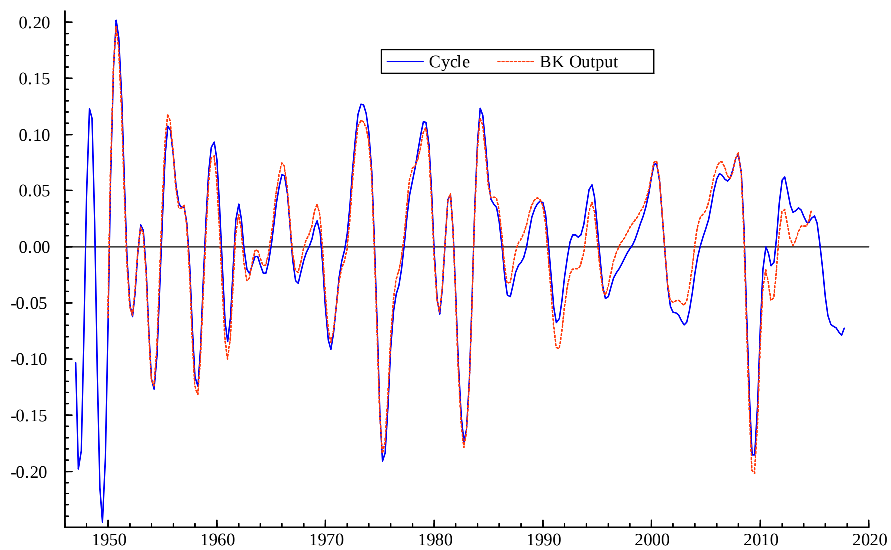
Estimated Band Pass filter for extracting cycle in Investment for n = 2 and 6 with Balanced form

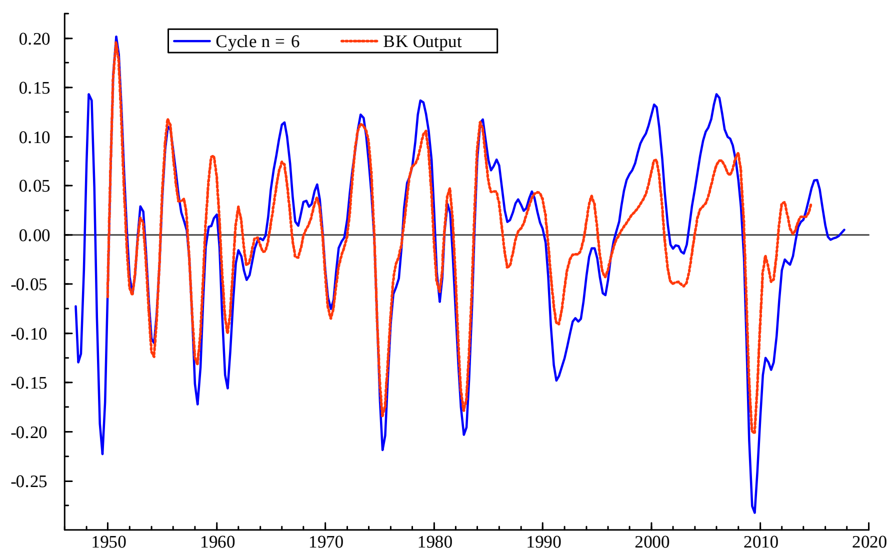
Even stronger link to band pass for orders above 2, more noise removal, smoother cycles

Better fit (for Investment, a highly cyclical series)

Estimated cycle in Investment (in logs) for the modelled ideal filter with n = 6, compared to the BK filter output.

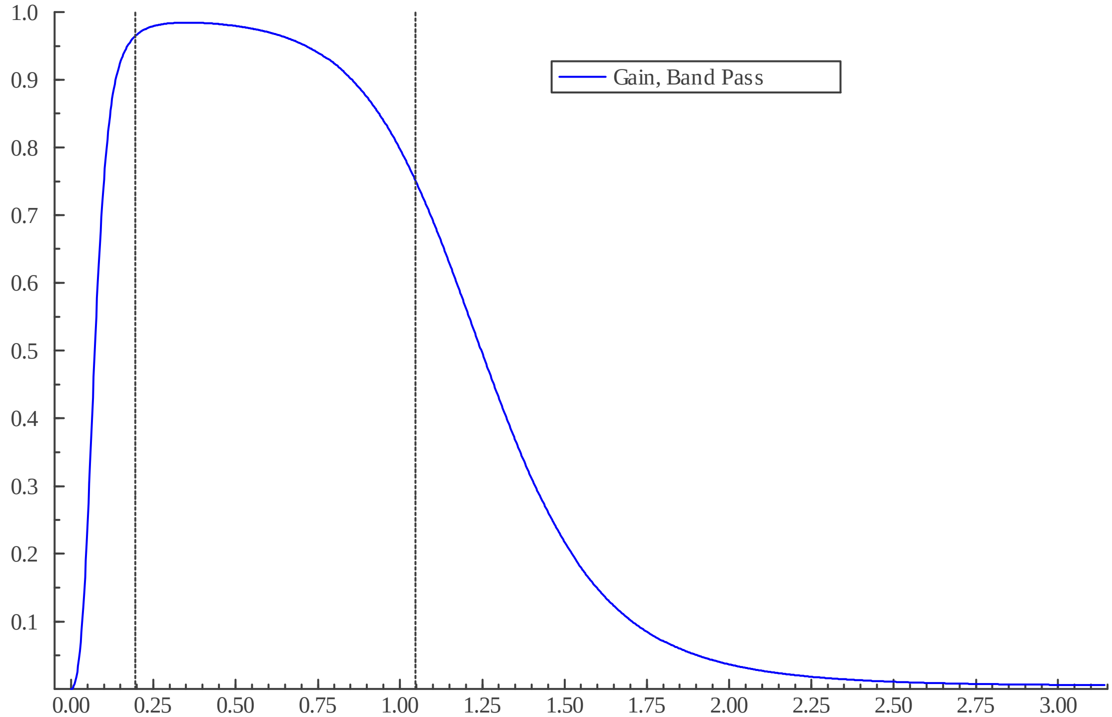
Results are generally close, some differences in second half, timing of cycles a bit different





Estimated cycle in Investment (in logs) for the adaptively modelled case with n = 6, compared to the BK filter output.

Substantial differences in amplitude, in path of cycle, note end where cycle reaching a trough and beginning to turn up



Estimated Band Pass filter for extracting the cycle in Investment for n = 6 with Balanced form, shown with ideal filter boundaries (dotted lines).

Substantial differences in estimated gain and ideal gain

Estimated gain admits more frequency components around left edge of band pass – this gives a stronger, more persistent cycle

Estimated gain also admits more components around right edge

In general, estimated gain is more inclusive

21 Empirical Results

Diagnostics extremely negative for ideal filter's underlying models

Substantial serial correlation in residuals

Ideal filter Models cannot be seriously entertained, especially for $n = 8$

Adaptively estimated models at higher orders perform well

Estimated gain and cycles differ significantly from ideal filter, by an amount that varies across series

For highly cyclical series, very high orders perform well, suggesting resonance in these cycles

In nearly cases, the first order cycle is outperformed by higher orders

22 Conclusions

Develop Range of Modelled Approximations to Ideal Filter

These Depend Critically on Higher Order Cycles

In case where ideal is appropriate, modelled representation has advantages over BK

Ideal Filter models do very poorly in terms of fit

This connection reveals inadequacy of ideal filter, how an adaptive gain should instead be used

Higher Order Cycles give smoother estimated cycles, generally and link with band-pass gets stronger as order incresaes

Also provide better fits for a range of time series of US economic activity

For highly cyclical series, the best index can be as high as 6 to 8

23 Future work

Higher order models applied to other kinds of time series - e.g. price series for housing, commodities, other sectors

Applied to exchage rates, financial series, energy series, output in countries other than U.S., Climate time series

Used in connection with other kinds of models, e.g. oil price determinants such as OPEC behavior

Forecasts of cycles

Used for turning point detection, real-time analysis

Multivariate - link housing cycle and business cycle, relate commodity price cycles to business cycles, different time series associated with El Nino/Southern Oscillation indices